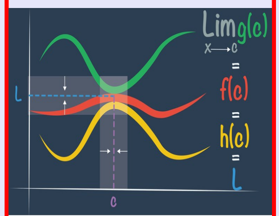
Calculus I

Lecture 8



Feb 19-8:47 AM

More on derivative:

$f(x) = (2x+1)^2$ Find $f'(x)$

Method I

$(A+B)^2 = A^2 + 2AB + B^2$

$f(x) = (2x)^2 + 2(2x)\cdot 1 + 1^2$

$= 4x^2 + 4x + 1$

$\boxed{f'(x) = 8x + 4} = 4(2x+1) = 2\cdot 2(2x+1)$

Method II

$f(x) = (2x+1)^2$

$= (2x+1)(2x+1)$

$f'(x) = 2(2x+1) + (2x+1)\cdot 2$

$= 4x + 2 + 4x + 2$

$= 8x + 4 = 4(2x+1) = 2\cdot 2(2x+1)$

Jan 15-10:01 AM

Chain Rule

If $y = f(u)$ and $u = g(x)$, then

$$y' = f'(u) \cdot u'$$

$$y' = f'(u) \cdot g'(x)$$

$f(x) = (\underbrace{2x+1}_{u})^2$ $\quad u^2$

$f'(x) = 2u \cdot u' = 2(2x+1) \cdot 2$

$= 2 \cdot 2(2x+1)$

Jan 15-10:07 AM

$f(x) = (\underbrace{x^2 - 3x + 5}_{u})^4$ $\quad u^4$

$f'(x) = 4u^3 \cdot u'$

$= 4(x^2 - 3x + 5)^3 \cdot (2x - 3)$

$f(x) = \left(\underbrace{\frac{x-3}{x+5}}_{u}\right)^{10}$ $\quad u^{10}$

$u' = \dfrac{1(x+5) - (x-3) \cdot 1}{(x+5)^2}$

$= \dfrac{\cancel{x}+5 - \cancel{x}+3}{(x+5)^2}$

$= \dfrac{8}{(x+5)^2}$

$f'(x) = 10u^9 \cdot u'$

$= 10\left(\dfrac{x-3}{x+5}\right)^9 \cdot \dfrac{8}{(x+5)^2}$

$= \dfrac{80(x-3)^9}{(x+5)^{11}}$

Jan 15-10:11 AM

$f(x) = \sin(\underbrace{x^5+5}_{u})$ $\quad \sin u$

$u \longrightarrow u' = 5x^4$

$f'(x) = \cos u \cdot u'$

$= \cos(x^5+5) \cdot 5x^4$

$\boxed{f'(x) = 5x^4 \cos(x^5+5)}$

$f(x) = (\underbrace{4x - x^2}_{u})^{100}$ $\quad f(u) = u^{100}$

$u' = 4 - 2x$ $\quad f'(u) = 100u^{99} \cdot u'$

$f'(x) = 100(4x - x^2)^{99} \cdot (4 - 2x)$

$= 100\,(\underbrace{4-2x}_{2(2-x)})(4x - x^2)^{99}$

$= 200\,(2-x)(4x - x^2)^{99}$

Jan 15-10:20 AM

$f(x) = \sqrt[3]{1 + \tan x}$

$f(x) = (\underbrace{1 + \tan x}_{u})^{1/3}$

$f(u) = u^{1/3}$ $\quad f'(u) = \frac{1}{3}u^{\frac{1}{3}-1} \cdot \boxed{u'}$

$f'(x) = \frac{1}{3}(1+\tan x)^{-2/3} \cdot (0 + \sec^2 x)$

$f'(x) = \frac{1}{3} \cdot \frac{1}{(1+\tan x)^{2/3}} \cdot \sec^2 x$

$= \frac{\sec^2 x}{3\sqrt[3]{(1+\tan x)^2}}$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Best Approach:

Start from outside, work your way inside.

Jan 15-10:26 AM

$$f(x) = \cos(\sin x^2)$$

$$f'(x) = -\sin(\sin x^2) \cdot \cos(x^2) \cdot 2x$$

$$= -2x \sin(\sin x^2) \cdot \cos x^2$$

$$f(x) = \tan(\underbrace{x \cos x}_{\text{Product}})$$

$$f'(x) = \sec^2(x\cos x) \cdot \overbrace{[1 \cdot \cos x + x \cdot (-\sin x)]}^{\text{Product Rule}}$$

$$= \sec^2(x\cos x) \cdot (\cos x - x\sin x)$$

Jan 15-10:33 AM

$$f(x) = (3x-1)^4 (2x+1)^3$$

Product

$$f'(x) = \frac{d}{dx}\left[(3x-1)^4\right] \cdot (2x+1)^3 + (3x-1)^4 \cdot \frac{d}{dx}\left[(2x+1)^3\right]$$

$$= 4(3x-1)^3 \cdot 3(2x+1)^3 + (3x-1)^4 \cdot 3(2x+1)^2 \cdot 2$$

$$= 12(3x-1)^3(2x+1)^3 + 6(3x-1)^4(2x+1)^2$$

$$= 6(3x-1)^3(2x+1)^2\left[2(2x+1) + (3x-1)\right]$$

$$\boxed{f'(x) = 6(3x-1)^3(2x+1)^2(7x+1)}$$

Jan 15-10:39 AM

$$f(x) = \cos^4(\sin^3(x^5)) = [\cos(\sin^3(x^5))]^4$$

$$f'(x) = 4\cos^3(\sin^3(x^5)) \cdot -\sin(\sin^3(x^5)) \cdot 3 \cdot \sin^2(x^5) \cdot \cos(x^5)\, 5x^4$$

$$= -60x^4 \cos^3(\sin^3 x^5) \cdot \sin(\sin^3 x^5) \cdot \sin^2 x^5$$

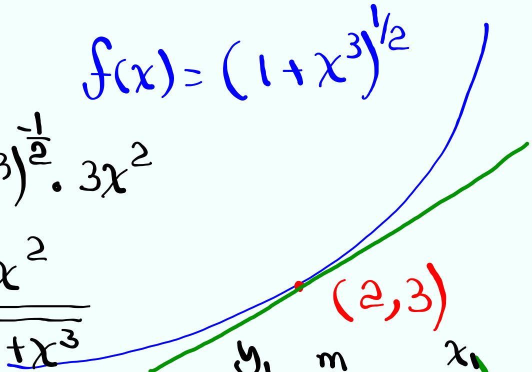
Jan 15-10:48 AM

Find equation of the tan. line to the graph of $f(x) = \sqrt{1+x^3}$ at $x = 2$.

$$f(x) = (1+x^3)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x^3)^{-\frac{1}{2}} \cdot 3x^2$$

$$= \frac{3x^2}{2\sqrt{1+x^3}}$$



$$m = f'(2)$$

$$= \frac{3(2)^2}{2\sqrt{1+2^3}}$$

$$= \frac{3 \cdot 4}{2 \cdot 3}$$

$$= \boxed{2}$$

$$y - \overset{y_1}{3} = \overset{m}{2}(x - \overset{x_1}{2})$$

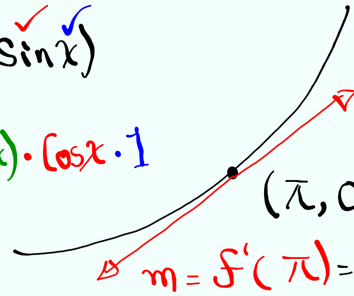
$$\boxed{y = 2x - 1}$$

Jan 15-10:55 AM

Find eqn of the tan. line to the graph of $f(x) = \sin(\sin x)$ at $x = \pi$.

$f(x) = \sin(\sin x)$

$f'(x) = \cos(\sin x) \cdot \cos x \cdot 1$



$(\pi, 0)$

$f(\pi) = \sin(\sin \pi) = \sin 0 = 0$

$m = f'(\pi) = \cos(\sin \pi) \cdot \cos \pi$

$= \cos 0 \cdot (-1)$

$= 1 \cdot (-1) = \boxed{-1}$

$y - y_1 = m(x - x_1)$

$y - 0 = -1(x - \pi)$

$\boxed{y = -x + \pi}$

Jan 15-11:01 AM

$f(1) = 7$, $f'(1) = 4$ $\quad h(x) = \sqrt{4 + 3f(x)}$

$h(x) = (4 + 3f(x))^{1/2}$

1) Find $h(1)$

$h(1) = \sqrt{4 + 3 \cdot f(1)}$

$= \sqrt{4 + 3 \cdot 7} = \sqrt{25} = \boxed{5}$

2) Find $h'(1)$

$h'(x) = \frac{1}{2}(4 + 3f(x))^{-1/2} \cdot (0 + 3 \cdot f'(x))$

$h'(x) = \dfrac{3f'(x)}{2\sqrt{4 + 3f(x)}}$

$h'(1) = \dfrac{3f'(1)}{2\sqrt{4 + 3f(1)}}$

$= \dfrac{3 \cdot 4}{2\sqrt{4 + 3 \cdot 7}} = \dfrac{3 \cdot 4}{2 \cdot 5} = \boxed{\dfrac{6}{5}}$

Jan 15-11:08 AM

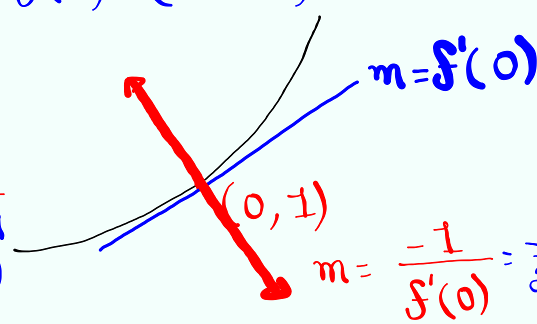
Find eqn of the normal line to the graph of $f(x) = (1+2x)^{10}$ at $x=0$.

$f(x) = (1+2x)^{10}$ ✓

$f'(x) = 10(1+2x)^{9} \cdot 2$

$f'(0) = 20(1+2(0))^{9}$

$= 20$



$m = \dfrac{-1}{f'(0)} = \dfrac{-1}{20}$

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{-1}{20}(x - 0)$

$\boxed{y = \frac{-1}{20}x + 1}$

Jan 15-11:15 AM

$f(x) = \dfrac{(2x-1)^3}{(3x+2)^5}$ Find $f'(x)$.

$$f'(x) = \frac{\frac{d}{dx}\left[(2x-1)^3\right]\cdot(3x+2)^5 - (2x-1)^3\cdot\frac{d}{dx}\left[(3x+2)^5\right]}{\left[(3x+2)^5\right]^2}$$

$$= \frac{3(2x-1)^2\cdot 2\cdot(3x+2)^5 - (2x-1)^3\cdot 5(3x+2)^4\cdot 3}{(3x+2)^{10}}$$

$$= \frac{6(2x-1)^2(3x+2)^5 - 15(2x-1)^3(3x+2)^4}{(3x+2)^{10}}$$

$$= \frac{3(2x-1)^2(3x+2)^4\left(2(3x+2) - 5(2x-1)\right)}{(3x+2)^{10}}$$

$$= \frac{3(2x-1)^2\cancel{(3x+2)^4}(-4x+9)}{(3x+2)^{\cancel{10}\,6}}$$

$$= \frac{3(2x-1)^2(-4x+9)}{(3x+2)^6}$$

Jan 15-11:20 AM

$y = \sqrt{x}$, Find y'.

$y = x^{1/2}$ $\quad y' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

New method:

$y = \sqrt{x}$

Square both Sides $\quad y^2 = x$

take the derivative of both Sides with respect to x $\quad \frac{d}{dx}[y^2] = \frac{d}{dx}[x]$

$2y \cdot \frac{dy}{dx} = 1$

isolate $\frac{dy}{dx}$ $\quad \frac{dy}{dx} = \frac{1}{2y}$

$\boxed{y' = \frac{1}{2\sqrt{x}}}$

Jan 15-11:30 AM

$y = \sqrt[3]{x^2}$

$y^3 = x^2$

$\frac{d}{dx}[y^3] = \frac{d}{dx}[x^2]$

$3y^2 \frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{3y^2}$

Jan 15-11:37 AM

$x = \sin y$ find $\dfrac{dy}{dx}$

$y = \sin^{-1} x$

$\dfrac{d}{dx}[x] = \dfrac{d}{dx}[\sin y]$

$1 = \cos y \cdot \dfrac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos y}}$$

Jan 15-11:40 AM

Find equation of the tan. line to the graph $x^2+y^2=25$ at $(3,4)$.

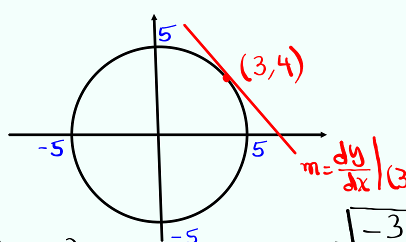
Is $(3,4)$ on the circle?

$3^2+4^2=25$

$9+16=25$

$25=25$

Yes



$m = \left.\dfrac{dy}{dx}\right|_{(3,4)} = \boxed{\dfrac{-3}{4}}$

$x^2+y^2=25$

$\dfrac{d}{dx}[x^2+y^2] = \dfrac{d}{dx}[25]$

$2x + 2y \cdot \dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-2x}{2y} = \dfrac{-x}{y}$

$y - y_1 = m(x - x_1)$

$y - 4 = \dfrac{-3}{4}(x-3)$

$y = \dfrac{-3}{4}x + \dfrac{9}{4} + 4$

$$\boxed{y = \frac{-3}{4}x + \frac{25}{4}}$$

Jan 15-11:43 AM

Find $\frac{dy}{dx}$ for $x^2 + xy - y^2 = 4$

$$\frac{d}{dx}\left[x^2 + xy - y^2\right] = \frac{d}{dx}[4]$$

$$\frac{d}{dx}[x^2] + \frac{d}{dx}[xy] - \frac{d}{dx}[y^2] = 0$$

$$2x + \frac{d}{dx}[x]\cdot y + x\cdot\frac{d}{dx}[y] - 2y\cdot\frac{dy}{dx} = 0$$

$$2x + 1\cdot y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$[x - 2y]\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x-2y}$$

$$= \frac{-(2x+y)}{-(2y-x)} = \boxed{\frac{2x+y}{2y-x}}$$

Jan 15-11:50 AM

Find $\frac{dy}{dx}$ for $x\sin y + y\sin x = 1$

$$\frac{d}{dx}\left[x\sin y + y\sin x\right] = \frac{d}{dx}[1]$$

$$\frac{d}{dx}[x\sin y] + \frac{d}{dx}[y\sin x] = 0$$

$$\frac{d}{dx}[x]\sin y + x\cdot\frac{d}{dx}[\sin y] + \frac{d}{dx}y\cdot\sin x + y\cdot\frac{d}{dx}[\sin x] = 0$$

$$1\cdot\sin y + x\cdot\cos y\cdot\frac{dy}{dx} + \frac{dy}{dx}\cdot\sin x + y\cdot\cos x = 0$$

$$(x\cos y + \sin x)\frac{dy}{dx} = -\sin y - y\cos x$$

$$\boxed{\frac{dy}{dx} = \frac{-\sin y - y\cos x}{x\cos y + \sin x}}$$

This method is called implicit Differentiation

use it when

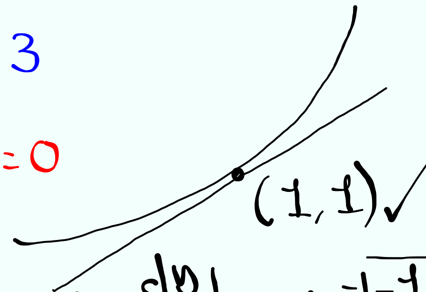
y cannot be explicity written in terms of x.

Jan 15-12:01 PM

find eqn of the tan. line to the graph of $x^2 + xy + y^2 = 3$ at $(1, 1)$.

$x^2 + xy + y^2 = 3$

$2x + 1 \cdot y + x \cdot \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$



$2(1) + 1 \cdot 1 + 1m + 2 \cdot 1 \cdot m = 0 \quad m = \frac{dy}{dx}\Big|_{(1,1)} = \boxed{-1}$

$2 + 1 + m + 2m = 0$

$3 + 3m = 0$

$\boxed{m = -1}$

$y - y_1 = m(x - x_1)$

$y - 1 = -1(x - 1)$

$\boxed{y = -x + 2}$

Jan 15-12:11 PM